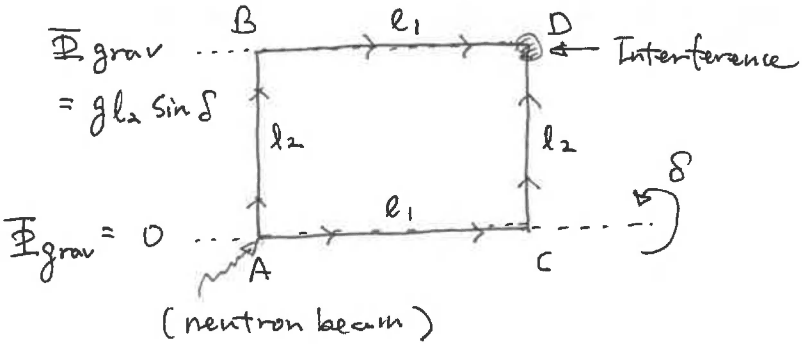
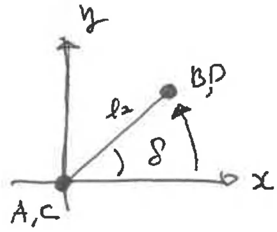




$\Rightarrow$ phase factor $= \exp\left[-\frac{i}{\hbar} m_n g l_2 \sin\delta \cdot T\right]$

Travel time along $\overline{BD}$

$$T = \frac{l_1}{v_n} \approx l_1 \Big/ \frac{\hbar}{m\lambdabar}$$

$\lambdabar$: de Broglie wavelength $= \frac{\hbar}{p} = \frac{\hbar}{m v_n}$

② Back to the EM fields : a charged particle in the EM-fields.

- Review on the classical Mechanics.

a charge (if it's electron, $e<0$)

i) Lagrangian : $L = \frac{1}{2} m \dot{\vec{x}}^2 - e\phi + \frac{e}{c}\dot{\vec{x}} \cdot \vec{A}(\vec{x},t)$

EOM : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{e}{c}A_i \qquad \frac{\partial L}{\partial x_i} = -e\frac{\partial \phi}{\partial x_i} + \sum_j \frac{e}{c}\dot{x}_j \frac{\partial A_j}{\partial x_i}$$

$$\Rightarrow \quad m\ddot{x}_i + \frac{e}{c}\left(\frac{\partial A_i}{\partial t} + \sum_j \frac{\partial A_i}{\partial x_j}\dot{x}_j\right) + e\frac{\partial \phi}{\partial x_i} - \sum_j \frac{e}{c}\dot{x}_j\frac{\partial A_j}{\partial x_i} = 0 .$$

$$m\ddot{x}_i = -e\underbrace{\left[\frac{\partial \phi}{\partial x_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right]}_{=\left(\nabla\phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)_i = (-\vec{E})_i} + \frac{e}{c}\sum_j\left[\dot{x}_j\frac{\partial A_j}{\partial x_i} - \dot{x}_j\frac{\partial A_i}{\partial x_j}\right]$$

$$= \dot{x}_j \,\varepsilon_{ijk}\,\underbrace{(\nabla\times\vec{A})_k}_{=\vec{B}} = (\dot{\vec{x}}\times\vec{B})_i$$

$$\Rightarrow \quad m\ddot{\vec{x}} = e\vec{E} + \frac{e}{c}\dot{\vec{x}}\times\vec{B}$$

The Lagrangian is verified.

ii) Hamiltonian $H(\vec{x},\vec{p};t) = \dot{\vec{x}}\cdot\vec{p} - L(\vec{x},\dot{\vec{x}};t)$

canonical momentum $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{e}{c}A_i$

kinematical momentum (mechanical) $\dot{x}_i = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right)$!

$$H = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right)p_i - \frac{1}{2}m\left[\frac{1}{m}\left(p_i - \frac{e}{c}A_i\right)\right]^2 + e\phi$$

$$-\frac{e}{c}\frac{1}{m}\left(p_i - \frac{e}{c}A_i\right)\cdot A_i \qquad \Big\| \ \sum_i \text{ is omitted.}$$

$$\Rightarrow H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}\right]^2 + e\phi \quad //$$

$\updownarrow$ correspondence.

- Quantum Mechanics

$$H = \frac{1}{2m}\left[\underset{\sim}{\vec{p}} - \frac{e}{c}\vec{A}\right]^2 + e\phi$$

This is an operator, too.

- EoM: (Heisenberg picture).

$$\frac{d\tilde{x}_i}{dt} = \frac{1}{i\hbar}[\tilde{x}_i, H] = \frac{1}{m}\left(\tilde{p}_i - \frac{e}{c}A_i\right)$$

$$\to m\frac{d\underset{\sim}{\vec{x}}}{dt} = \underset{\sim}{\vec{p}} - \frac{e}{c}\vec{A} \equiv \vec{\pi} \quad (\text{Kinematic momentum}).$$

So far, so good. Nothing's strange!

But, it's QM: There should be something strange! //

yes, it starts from here.

- Commutation relation

$[\tilde{x}_i, \tilde{x}_j] = 0$, $[\tilde{p}_i, \tilde{p}_j] = 0$, but $[\tilde{p}_i, A_j] = -i\hbar\frac{\partial A_j}{\partial x_i} \neq 0$. !

$$\Rightarrow [\pi_i, \pi_j] = [\tilde{p}_i - \tfrac{e}{c}A_i, \tilde{p}_j - \tfrac{e}{c}A_j]$$

$$= -\frac{e}{c}[\tilde{p}_i, A_j] - \frac{e}{c}[A_i, \tilde{p}_j] = \frac{e}{c}\Big[[\tilde{p}_j, A_i] - [\tilde{p}_i, A_j]\Big]$$

$$= \frac{i\hbar e}{c}[\partial_i A_j - \partial_j A_i] = \frac{i\hbar e}{c}\epsilon_{ijk}B_k$$

$$\therefore \boxed{[\pi_i, \pi_j] = \frac{i\hbar e}{c}\epsilon_{ijk}B_k.} \ \star$$

Classical Mechanics vs. Quantum Mechanics

$$m\frac{d^2\vec{x}}{dt^2} = e\vec{E} + \frac{e}{c}\left(\frac{d\vec{x}}{dt}\times\vec{B}\right) \longrightarrow \text{EOM}: \quad m\frac{d^2\vec{x}}{dt^2} = \frac{d\vec{\pi}}{dt} = \frac{1}{i\hbar}[\vec{\pi}, H]$$

$$= e\vec{E} + \frac{e}{c}\left(\frac{d\vec{x}}{dt}\times\vec{B}\right) : \Longleftrightarrow \text{classical}$$

$$+ \underline{\frac{ie\hbar}{2c}\nabla\times\vec{B}} \,! : \text{"Quantum effect"}$$

(absent when $\vec{B}$ = const.)

continuous. .... Energy ... Quantized.

(set by any initial conditions)

ex. 2DEG (2D electron gas)

$\vec{B} = B\hat{z}$, $\vec{E} = 0$

$$H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2)$$

$\Leftarrow$ just like a simple harmonic oscillator!

"Landau Levels"

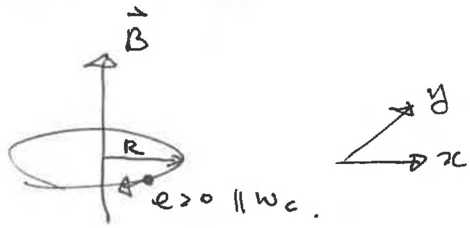
$$\longrightarrow E = \hbar\omega_c\left(n+\tfrac{1}{2}\right), \quad n = 0, 1, 2, \cdots$$

$\parallel$ $\omega_c = \dfrac{eB}{mc}$: cyclotron freq.

ex. 2D,

$\vec{B} = B\hat{z}$, $\vec{E} = 0$

.... Motion (orbit) ... Quantized.



$$\to \frac{d\Pi_x}{dt} = \omega_c\Pi_y, \quad \frac{d\Pi_y}{dt} = -\omega_c\Pi_x$$

$$\Rightarrow \tilde{x}(t) = \tilde{x}_0 - \frac{1}{m\omega_c}\Pi_y(t)$$

$$\tilde{y}(t) = \tilde{y}_0 + \frac{1}{m\omega_c}\Pi_x(t)$$

$\hookrightarrow$ operators, but constants of motion

!!! $[\tilde{x}_0, \tilde{y}_0] = -i\, l_B^2$ $\parallel$ $l_B = \sqrt{\hbar/m\omega_c}$: magnetic length.

R = continuous ;
it grows continuously
as the energy grows.

But, $[H, \tilde{x}_0] = 0$
$[H, \tilde{y}_0] = 0$ $\rightarrow$ $\tilde{x}_0, \tilde{y}_0$ are time-invariant.

$$W_c = \frac{eB}{mc} \quad \text{(CGS unit)}.$$

$$\Rightarrow R^2 = [x(t) - x_0]^2 + [y(t) - y_0]^2$$

$$= \frac{2}{m\omega_c^2} H \quad !!!$$

$$\langle R^2\rangle_n = (2n+1)\, l_B^2 : \text{quantized!}$$

$\to$ The radius grows as $\sqrt{n}$.

- Gauge invariance ; "Gauge Transformation"

For $\vec{B} = B\hat{z}$, $\vec{A}_1 = (-\frac{B}{2}y, \frac{B}{2}x, 0)$

and $\vec{A}_2 = (-By, 0, 0)$,

Both give the same $\vec{B} = \nabla \times \vec{A}_{1,2} = B\hat{z}$.

There's some freedom to choose "gauge".

→ Gauge transformation $\vec{A}_2 = \vec{A}_1 - \nabla\left(\frac{Bxy}{2}\right)$, here.

In general, $\vec{A}' = \vec{A} + \nabla\Lambda$.

Q. How does this gauge transformation changes the quantum dynamics of a particle in $\vec{B}$?

- Expectation values ~~are~~ have to be! invariant.

Q. Is there such $\mathcal{G}$ op. ?

$$\langle\alpha|\tilde{x}_i|\alpha\rangle = \langle\alpha'|\tilde{x}_i'|\alpha'\rangle \;\cdots(*) \qquad \Big\| \quad |\alpha'\rangle = \boxed{\mathcal{G}}|\alpha\rangle$$

$$\langle\alpha|\Pi_i|\alpha\rangle = \langle\alpha'|\Pi_i'|\alpha'\rangle \;\cdots(**)$$

(primed is associated with $\vec{A}'$, unprimed is with $\vec{A}$.)

- The form of the Schrödinger eq. has to be invariant.

$$\rightarrow \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2|\alpha,t\rangle = i\hbar\frac{\partial}{\partial t}|\alpha,t\rangle$$

and

$$\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}'\right)^2|\alpha',t\rangle = i\hbar\frac{\partial}{\partial t}|\alpha',t\rangle.$$

$\Big\|$ $\vec{A} \equiv \vec{A}(\vec{x})$, $\Lambda \equiv \Lambda(\vec{x})$, $\phi = 0$.

In $\vec{x}$- representation,

$$\frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x})\right)^2\langle\vec{x}|\alpha,t\rangle = i\hbar\frac{\partial}{\partial t}\langle\vec{x}|\alpha,t\rangle.$$

↑ all effects of a $\vec{B}$-field are here!

Maybe, we can treat this equation as if it's about a particle in no $\vec{B}$-field by $\nabla_{new} = \nabla - \frac{ie}{\hbar c}\vec{A}$.

So, one may think that the continuity equation *

may be written as

$$\cancel{\frac{\partial \rho}{\partial t} + \left(\nabla - \frac{ie}{\hbar c}\vec{A}\right)\cdot\vec{J} = 0} .$$

But, THIS IS WRONG !

|| $\rho = |\psi|^2$

→ $\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0$ has to be invariant.

→ This is divergence, not an operator. (physically)

Instead, $\vec{J} = \frac{\hbar}{m}\,\mathrm{Im}[\psi^* \nabla_{new} \psi]$

← This is "momentum".

$$= \frac{\hbar}{m}\,\mathrm{Im}[\psi^*\nabla\psi] - \underbrace{\frac{e}{mc}\vec{A}\,|\psi|^2}_{\text{an extra term}} .$$

!

→ So, the current density depends on the choice of $\vec{A}$! ?

∴ it shouldn't ! ☆

If we use a general form of the wave function,

$$\psi(x) \equiv \sqrt{\rho(x)}\,\exp\left[\frac{i}{\hbar}\overset{\text{"phase"}}{S}\right],$$

$$\vec{J} = \frac{\rho}{m}\left(\nabla S - \frac{eA}{c}\right).$$

|| If $A=0$, $\vec{J} = \frac{\rho}{m}\overset{m\vec{v}}{\nabla S}$

$= \frac{\hbar}{m}\,\mathrm{Im}[\psi^*\nabla\psi]$.

⇒ When $\vec{A} \to \vec{A}' = \vec{A} + \nabla\Lambda$,

$S \to S' = S + \frac{e}{c}\Lambda$, → This makes $\vec{J}$ gauge-invariant. !

meaning that $\psi'(x) = \sqrt{\rho(x)}\,\exp\left[\frac{i}{\hbar}S\right]\exp\left[\frac{ie}{\hbar c}\Lambda\right]$

$$= \exp\left[\frac{ie}{\hbar c}\Lambda\right]\psi(x)$$

If we set

So $|\alpha'\rangle = \mathcal{G}|\alpha\rangle$, $\mathcal{G} = \exp\left[\frac{ie}{\hbar c}\Lambda(\vec{x})\right]$. //

You can check if (*) and (**) are hold.

(*) ... $g^{+} \underset{\sim}{\vec{x}}\, g = \underset{\sim}{\tilde{x}}' = \underset{\sim}{\tilde{x}}$: obvious. (The position operator is the position operator.)

(**) ... $g^{+} (\underset{\sim}{\vec{p}} - \frac{e}{c}\vec{A} - \frac{e}{c}\nabla\Lambda)\, g = \underset{\sim}{\tilde{p}} - \frac{e}{c}\vec{A}$

· $\vec{A}(\underset{\sim}{\vec{x}})$, $\Lambda(\underset{\sim}{\vec{x}})$ commute with $g(\underset{\sim}{\vec{x}})$.

· $$e^{-\frac{ie\Lambda}{\hbar c}} \underset{\sim}{\vec{p}}\, e^{\frac{ie\Lambda}{\hbar c}} = e^{\frac{-ie\Lambda}{\hbar c}} [\underset{\sim}{\vec{p}}, e^{\frac{ie\Lambda}{\hbar c}}] + \underset{\sim}{\vec{p}}$$

$$= e^{\frac{-ie\Lambda}{\hbar c}} (-i\hbar\nabla)\, e^{\frac{ie\Lambda}{\hbar c}} + \underset{\sim}{\vec{p}}.$$

$$= \underset{\sim}{\vec{p}} + \frac{e}{c}\nabla\Lambda(\underset{\sim}{\vec{x}})$$

$$\Rightarrow g^{+}(\underset{\sim}{\vec{p}} - \frac{e}{c}\vec{A} - \frac{e}{c}\nabla\Lambda)\, g = \underset{\sim}{\vec{p}} + \cancel{\frac{e}{c}\nabla\Lambda} - \frac{e}{c}\vec{A} - \cancel{\frac{e}{c}\nabla\Lambda}$$

(try with H by yourself : $g^{+}Hg$) #.

Indeed, $|\alpha'\rangle = \exp\left[\frac{ie}{\hbar c}\Lambda(\underset{\sim}{\vec{x}})\right]|\alpha\rangle$.

: The gauge transformation, $\vec{A} \to \vec{A} + \nabla\Lambda$,

introduces an extra phase factor ! in $\psi(x)$;

by changing $\vec{A}$ ($\vec{B}$ = same), one may expect some interferences due to the difference bet. the accumulated phases

• Example 1: The Aharonov-Bohm effect.

" $\vec{B}=0$ does not necessarilly mean $\vec{A}=0$. "